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Errors in Cubic Spline Interpolation

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SUMMARY

This paper deals with the problem of finding error bounds for cubic spline interpolation of functions of the class $C^4[a, b]$, and $C^5[a, b]$, by examining a relationship between cubic spline interpolation and piecewise cubic Hermitian interpolation.

The method also gives an indication of what happens, in the case of almost uniform meshes, especially if the approximated function is in the class $C^5[a, b]$.

Comparison is made with recent work carried out by K. E. Atkinson [3], in dealing with natural cubic spline interpolation.

1. Introduction

Let h denote a subdivision of the interval $[a, b]$ of the real x-axis

$$h = \{a = x_0 < x_1 < \ldots < x_n = b\} . \tag{1.1}$$

A cubic spline, or spline on the mesh h is defined by the following relations

$$\begin{aligned} &y_h(x) \in C^2[a, b] \\ &y_h(x) = y_i + p_i(x-x_i) + \tfrac{1}{2}q_i(x-x_i)^2 + \tfrac{1}{6}r_i(x-x_i)^3 \\ &\text{for } x_i \leqq x \leqq x_{i+1}, \quad 0 \leqq i \leqq n-1. \end{aligned} \tag{1.2}$$

The class of all splines on the mesh h will be denoted by Y_h.

From the continuity requirements in (1.2) the following set of linear relations between the polynomial coefficients can be derived

$$\begin{aligned} &\text{a.} \quad \lambda_j p_{j-1} + 2p_j + \mu_j p_{j+1} = 3\{\mu_j \Delta y_j + \lambda_j \Delta y_{j-1}\}, \quad \text{for } 1 \leqq j \leqq n-1, \\ &\text{b.} \quad q_0 = \frac{6\Delta y_0 - 4p_0 - 2p_1}{h_1}, \quad q_n = -\frac{6\Delta y_{n-1} - 4p_n - 2p_{n-1}}{h_n} \\ &\text{c.} \quad q_j = -q_{j-1} + \frac{2}{h_j}\{p_j - p_{j-1}\}, \qquad 1 \leqq j \leqq n-1 \\ &\text{d.} \quad r_j = \frac{q_{j+1} - q_j}{h_{j+1}}, \qquad 0 \leqq j \leqq n-1 \end{aligned} \tag{1.3}$$

where

$$h_j = x_j - x_{j-1}, \quad \lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}, \quad \mu_j = 1 - \lambda_j,$$

$$\Delta y_j = \frac{y_{j+1} - y_j}{h_{j+1}}.$$

The set (1.3) consists of $3n$ equations between $4n+3$ coefficients.
It follows that if $n+3$ independent extra relations are given, a spline is uniquely determined.

We can define several types of splines, interpolating a given function $f(x)$ in the $n+1$ mesh-points x_i, and satisfying two extra so-called end conditions.

The first of these interpolating splines, denoted by $y_h(f;x)$, is defined by

$$\begin{aligned} &y_h(f;x)\in Y_h\,. \\ &y_h(f;x_i)=f(x_i)\,, \qquad 0\leqq i\leqq n\,. \\ &y_h'(f;x_i)=f'(x_i)\,, \qquad i=0 \text{ and } i=n\,. \end{aligned} \tag{1.4}$$

All coefficients p_j can be determined by solving the system (1.3.a) together with the conditions: $p_0=f'(x_0)$, $p_n=f'(x_n)$. The system (1.3.b,c) yields the values of the q_j, and finally (1.3.d) yields the values of the r_j.

The spline interpolation $y_h(f;x)$ will be referred to as the optimal interpolating spline, since it has the best approximation property

$$\int_a^b \{y_h''(f;x)-f''(x)\}^2\,dx \leqq \int_a^b \{z''(x)-f''(x)\}^2\,dx \quad \text{for all } z\in Y_h\,. \tag{1.5}$$

Another type of spline interpolation, denoted by $y_h^*(f;x)$, is defined by

$$\begin{aligned} &y_h^*(f;x)\in Y_h \\ &y_h^*(f;x_i)=f(x_i), \qquad 0\leqq i\leqq n\,. \\ &y_h^{*\prime\prime}(f;x_i)=f''(x_i), \qquad i=0 \text{ and } i=n\,. \end{aligned} \tag{1.6}$$

This spline interpolation can be computed by substituting $q_0=f''(x_0)$, $q_n=f''(x_n)$ in (1.3.b), and solving the p_j from the system (1.3.a,b); (1.3.c) and (1.3.d) serve to determine $q_1, q_2, \ldots, q_{n-1}$ and $r_0, r_1, \ldots, r_{n-1}$ by substitution.

A third type of spline interpolation is the so-called natural spline interpolation, defined by

$$\begin{aligned} &\tilde{y}_h(f;x)\in Y_h \\ &\tilde{y}_h(f;x_i)=f(x_i), \qquad 0\leqq i\leqq n\,. \\ &\tilde{y}_h(f;x_i)=0, \qquad i=0 \text{ and } i=n\,. \end{aligned} \tag{1.7}$$

This spline has the minimum norm property

$$\int_a^b \{\tilde{y}_h''(f;x)\}^2\,dx \leqq \int_a^b \{z''(f;x)\}^2\,dx \tag{1.8}$$

for all interpolating $z(f;x)\in C^2[a,b]$.

Defining the splines $l_0(x)$ and $l_n(x)$ by

$$\begin{aligned} &l_0\in Y_h, \quad l_n\in Y_h\,, \\ &l_0(x_i)=l_n(x_i)=0, \qquad 0\leqq i\leqq n\,. \\ &l_0''(x_0)=l_n''(x_n)=1 \\ &l_0''(x_n)=l_n''(x_0)=0\,. \end{aligned} \tag{1.9}$$

will show that the following relation between $y_h^*(f;x)$ and $\tilde{y}_h(f;x)$ holds

$$\tilde{y}_h(f;x)=y_h^*(f;x)-f''(x_0)l_0(x)-f''(x_n)l_n(x)\,. \tag{1.10}$$

The error $E_h(f;x)$ of a spline interpolation is defined by

$$E_h(f;x)=f(x)-y_h(f;x)\,. \tag{1.11}$$

Now let

$$\|h\| = \max_{1\leqq j\leqq n} h_j, \text{ and } \|g(x)\| = \max_{a\leqq x\leqq b} |g(x)|\,,$$

then if $f\in C^4[a,b]$, the error satisfies the following inequality

$$\begin{aligned} &\|E_h(f;x)\|\leqq K(f)\cdot\|h\|^4 \\ &\|E_h^*(f;x)\|\leqq K^*(f)\|h\|^4\,. \end{aligned} \tag{1.12}$$

The purpose of this paper is to find the best possible values for the constants $K(f)$ and $K^*(f)$.

In [2] and [3] an integral representation for the error is used in order to obtain error estimates.

$$E_h(f;x)=\int_a^b K(x,t)f^{\mathrm{IV}}(t)\,dt$$
$$E_h^*(f;x)=\int_a^b K^*(x,t)f^{\mathrm{IV}}(t)\,dt \tag{1.13}$$

where $K(x,t)$ and $K^*(x,t)$ are functions of the class $C^2[a,b]$ with respect to x and t.

Obviously the following relations hold

$$|E_h(f;x)|\leqq\int_a^b |K(x,t)|\,dt\cdot\|f^{\mathrm{IV}}\|$$
$$|E_h^*(f;x)|\leqq\int_a^b |K(x,t)|\,dt\cdot\|f^{\mathrm{IV}}\|\,. \tag{1.14}$$

Atkinson [3] derives the following bounds for the integral of $|K^*(x,t)|$, in cases of uniform meshes

$$.00156\|h\|^2(x-x_i)(x_{i+1}-x)<\int_a^b |K^*(x,t)|\,dt<$$
$$<1.18\|h\|^2(x-x_i)(x_{i+1}-x),\qquad x_i\leqq x\leqq x_{i+1},\ 0\leqq i\leqq n-1\,. \tag{1.15}$$

An improved version of this result will be given in section III.

2. Relation between Cubic Spline Interpolation and Piecewise Cubic Hermitian Interpolation

The classical piecewise cubic Hermitian interpolation can be defined by the following relations

$$\begin{aligned}&y_H(f;x)\in C^1[a,b]\\&y_H(f;x)=y_i+a_i(x-x_i)+\tfrac{1}{2}b_i(x-x_i)^2+\tfrac{1}{6}c_i(x-x_i)^3\\&\text{for } x_i\leqq x\leqq x_{i+1},\quad 0\leqq i\leqq n-1\\&\left.\begin{aligned}y_H(f;x_i)&=f(x_i)\\y_H'(f;x_i)&=f'(x_i)\end{aligned}\right\}\quad 0\leqq i\leqq n\,.\end{aligned} \tag{2.1}$$

Therefore the piecewise cubic Hermitian interpolation is a piecewise cubic function, having a continuous first derivative on $[a,b]$. Furthermore, if $f\in C^4[a,b]$, the error can be expressed as follows

$$E_H(f;x)=f(x)-y_H(f;x)=\frac{(x-x_i)^2(x-x_{i+1})^2}{4!}f^{\mathrm{IV}}(\xi_i)$$
$$\text{for } x_i\leqq x\leqq x_{i+1},\quad x_i\leqq\xi_i\leqq x_{i+1},\quad 0\leqq i\leqq n-1\,. \tag{2.2}$$

which is a classical result.

Therefore the order of the approximation is $O(\|h\|^4)$, which is in agreement with the errors $E_h(f,x)$ and $E_h^*(f;x)$ of the spline interpolations.

The above description shows that a similarity exists between cubic spline interpolation and piecewise cubic Hermitian interpolation.

Let us denote the difference of the two approximations by $z(x)$ in order to examine this similarity

$$z(x)=y_H(f;x)-y_h(f;x)=E_h(f;x)-E_H(f;x) \tag{2.3}$$

and similarly:

$$z^*(x) = y_H(f; x) - y_h^*(f; x) = E_h^*(f; x) - E_H(f; x). \tag{2.4}$$

Then it follows from (1.12) and (2.2)

$$\|z\| = O(\|h\|^4), \quad \|z^*\| = O(\|h\|^4), \quad \text{if } f \in C^4[a, b]. \tag{2.5}$$

Obviously $z(x)$ and $z^*(x)$ are piecewise cubics, satisfying the relations

$$\begin{aligned}
&z(x) \in C^1[a, b], \quad z^*(x) \in C^1[a, b].\\
&z(x_i) = 0, \quad z^*(x_i) = 0, \quad \text{for } 0 \leqq i \leqq n.\\
&z'(x_i) = E_h'(f; x_i) - E_H'(f; x_i) = E_h'(f; x_i), \quad \text{for } 0 \leqq i \leqq n,\\
&z^{*\prime}(x_i) = E_h^{*\prime}(f; x_i) - E_H'(f; x_i) = E_h^{*\prime}(f; x_i), \quad \text{for } 0 \leqq i \leqq n.
\end{aligned} \tag{2.6}$$

Now define a set of functions $Q_i(x)$ as follows

$$Q_i(x) = \begin{cases} (x - x_i)\left(\dfrac{x - x_{i+1}}{h_{i+1}}\right)^2, & x_i \leqq x \leqq x_{i+1},\ 0 \leqq i \leqq n-1, \\[2ex] (x - x_i)\left(\dfrac{x - x_{i-1}}{h_i}\right)^2, & x_{i-1} \leqq x \leqq x_i,\ 1 \leqq i \leqq n. \\[2ex] 0 & x \leqq x_{i-1},\ x \geqq x_{i+1}. \end{cases} \tag{2.7}$$

It follows that $Q_i(x)$ is a piecewise cubic function belonging to $C^1[a, b]$, and satisfying the relations

$$\begin{aligned}
&Q_i(x_j) = 0\\
&Q_i'(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad \text{for } 0 \leqq i \leqq n,\ 0 \leqq j \leqq n.
\end{aligned} \tag{2.8}$$

By differentiating (2.7) two times with respect to x, we get

$$\begin{aligned}
&Q_i''(x_j+) = -\frac{2}{h_{j+1}}\{2\delta_{ij} + \delta_{i,j+1}\}\\
&\text{for } 0 \leqq i \leqq n,\ 0 \leqq j \leqq n-1\\
&Q_i''(x_j-) = \frac{2}{h_j}\{2\delta_{ij} + \delta_{i,j-1}\}\\
&\text{for } 0 \leqq i \leqq n,\ 1 \leqq j \leqq n.
\end{aligned} \tag{2.9}$$

Now from (2.8) and (2.6) it follows that $z(x)$ and $z^*(x)$ can be expressed in terms of the functions $Q_i(x)$

$$\begin{aligned}
z(x) &= \sum_{i=0}^{n} z_i' Q_i(x), \qquad z_i' = z'(x_i)\\
z^*(x) &= \sum_{i=0}^{n} z_i^{*\prime} Q_i(x), \qquad z_i^{*\prime} = z^{*\prime}(x_i).
\end{aligned} \tag{2.10}$$

From (2.9) it is obvious that $z(x)$ and $z^*(x)$ have second derivatives which in general are discontinuous in the mesh-points. Furthermore using the definitions (2.3) and (2.4) we find the following relations

$$\begin{aligned}
&\delta z''(x_i) = \delta E_h''(f; x_i) - \delta E_H''(f; x_i) = -\delta E_H''(f; x_i)\\
&\delta z^{*\prime\prime}(x_i) = -\delta E_H''(f; x_i) \qquad \text{for } 1 \leqq i \leqq n-1
\end{aligned} \tag{2.11}$$

where $\delta g(x) = g(x+) - g(x-)$.

Substituting (2.9) and (2.10) in (2.11), we find the following $n-1$ relations between the z_i' and $\delta E_H''(f; x_i)$

$$\text{a.}\quad \lambda_i z_{i-1}' + 2z_i' + \mu_i z_{i+1}' = +\tfrac{1}{2}\frac{h_i h_{i+1}}{h_i + h_{i+1}}\,\delta E_H''(f; x_i)$$

$$\text{b.}\quad \lambda_i z_{i-1}^{*\prime} + 2z_i^{*\prime} + \mu_i z_{i+1}^{*\prime} = \tfrac{1}{2}\frac{h_i h_{i+1}}{h_i + h_{i+1}}\,\delta E_H''(f; x_i) \tag{2.12}$$

where λ_i and μ_i have the meaning as defined in (1.3).

Furthermore we find from (2.3)

$$z'(x_0) = z'(x_n) = 0 \tag{2.13}$$

and from 2.4)

$$\begin{aligned} z^{*\prime\prime}(x_0) &= E_h^{*\prime\prime}(f; x_0) - E_H''(f; x_0) = -E_H''(f; x_0) \\ z^{*\prime\prime}(x_n) &= E_h^{*\prime\prime}(f; x_n) - E_H''(f; x_n) = -E_H''(f; x_n), \end{aligned} \tag{2.14}$$

from which it follows by substitution of (2.9) and (2.10)

$$\begin{aligned} 2z_0^{*\prime} + z_1^{*\prime} &= +\tfrac{1}{2}h_1 E_H''(f; x_0) \\ 2z_n^{*\prime} + z_{n-1}^{*\prime} &= -\tfrac{1}{2}h_n E_H''(f; x_n). \end{aligned} \tag{2.15}$$

The system (2.12.a) together with (2.13), yields a uniquely solvable system of equations for the z_j' if $\delta E_H''(f; x_i)$ are known, and so does the system (2.12.b) together with (2.15) for the $z_j^{*\prime}$.

In order to obtain an expression for $\delta E_H''(f; x_i)$ we examine Taylor's expansion for $f(x)$ in the neighbourhood of $x = x_i$

$$\begin{aligned} f(x) &= f(x_i) + (x-x_i)f'(x_i) + \tfrac{1}{2}(x-x_i)^2 f''(x_i) + \tfrac{1}{6}(x-x_i)^3 f'''(x_i) \\ &\quad + \int_{x_i}^{x} \frac{(x-t)^3}{6} f^{\mathrm{IV}}(t)\,dt \\ &= S_i(x) + E_i(x). \end{aligned} \tag{2.16}$$

The piecewise cubic Hermitian interpolation of $f(x)$ can be written as

$$y_H(f; x) = y_H(S_i; x) + y_H(E_i; x), \tag{2.17}$$

and since $S_i(x)$ is its own piecewise cubic Hermitian interpolation this can be reduced

$$y_H(f; x) = S_i(x) + y_H(E_i; x). \tag{2.18}$$

Hence it follows for $E_H(f; x)$:

$$\begin{aligned} E_H(f; x) &= S_i(x) + E_i(x) - S_i(x) - y_H(E_i; x) \\ &= E_i(x) - y_H(E_i; x) = E_H(E_i; x). \end{aligned} \tag{2.19}$$

Now since $E_i(x_i) = E_i'(x_i) = 0$, the following relations can easily be verified

$$\begin{aligned} y_H(E_i; x) &= E_i(x_{i+1})\left\{\frac{x - x_i - Q_i(x) - Q_{i+1}(x)}{h_{i+1}}\right\} + \\ &\quad + E_i'(x_{i+1})Q_{i+1}(x), \quad \text{for } x_i \leqq x \leqq x_{i+1} \\ y_H(E_i; x) &= E_i(x_{i-1})\left\{\frac{x_i - x + Q_i(x) + Q_{i-1}(x)}{h_i}\right\} + \\ &\quad + E_i'(x_{i-1})Q_{i-1}(x), \quad \text{for } x_{i-1} \leqq x \leqq x_i. \end{aligned} \tag{2.20}$$

Differentiating (2.20) two times with respect to x and substituting $x = x_i$, we find by making use of (2.9)

$$\begin{aligned} y_H''(E_i; x_i+) &= \frac{2}{h_{i+1}^2}\{3E_i(x_{i+1}) - E_i'(x_{i+1})h_{i+1}\} \\ y_H''(E_i; x_i-) &= \frac{2}{h_i^2}\{3E_i(x_{i-1}) + E_i'(x_{i-1})h_i\}\,. \end{aligned} \tag{2.21}$$

Substituting the integral expression for $E_i(x)$, given by (2.16), we can reduce (2.21) to the following form

$$\begin{aligned} y_H''(E_i; x_i+) &= \int_{x_i}^{x_{i+1}} \frac{(x_{i+1}-t)^2}{h_{i+1}^2}(x_i-t)f^{\mathrm{IV}}(t)\,dt \\ y_H''(E_i; x_i-) &= \int_{x_i}^{x_{i-1}} \frac{(x_{i-1}-t)^2}{h_i^2}(x_i-t)f^{\mathrm{IV}}(t)\,dt\,. \end{aligned} \tag{2.22}$$

Now if we substitute these results in (2.19) and bear in mind that $E_i''(x_i)=0$, we finally arrive at the following result

$$\begin{aligned} E_H''(f; x_i+) &= \int_{x_i}^{x_{i+1}} Q_i(t)f^{\mathrm{IV}}(t)\,dt, \quad 0 \leqq i \leqq n-1 \\ E_H''(f; x_i-) &= \int_{x_i}^{x_{i-1}} Q_i(t)f^{\mathrm{IV}}(t)\,dt, \quad 1 \leqq i \leqq n\,. \end{aligned} \tag{2.23}$$

from which it follows

$$\delta E_H''(f; x_i) = \int_{x_{i-1}}^{x_{i+1}} Q_i(t)f^{\mathrm{IV}}(t)\,dx = \int_a^b Q_i(t)f^{\mathrm{IV}}(t)\,dx \quad \text{for } 1 \leqq i \leqq n-1\,. \tag{2.24}$$

With the aid of (2.23) and (2.24), we can now reduce the equations (2.12), (2.13) and (2.14) to the following form

$$\begin{aligned} &\text{a.} \quad \begin{cases} \lambda_j z_{j-1}' + 2z_j' + \mu_j z_{j+1}' = R_j, & 1 \leqq j \leqq n-1 \\ z_0' = z_n' = 0\,. \end{cases} \\ &\text{b.} \quad \begin{cases} \lambda_j z_{j-1}^{*\prime} + 2z_j^{*\prime} + \mu_j z_{j+1}^{*\prime} = R_j & 0 \leqq j \leqq n \\ \lambda_0 = \mu_n = 0, \quad \mu_0 = \lambda_n = 1\,. \end{cases} \end{aligned} \tag{2.25}$$

with

$$R_j = \tfrac{1}{2}\nu_j \int_a^b Q_j(t)f^{\mathrm{IV}}(t)\,dt\,, \qquad 0 \leqq j \leqq n$$

$$\nu_j = \mu_j h_{j+1}\,, \qquad 0 \leqq j \leqq n-1$$

$$\nu_n = \lambda_n h_n = h_n\,.$$

The systems (2.25.a) and (2.25.b) have the same structure as the defining equations (1.3), and can also be derived directly from the defining equations. This is done by substituting $p_i = y_h'(f; x_i) = f'(x_i) - z_i'$ in (1.3.a). Applying a Taylor series expansion with integral remainder to the resulting right-hand members, yields again the systems (2.25).

Although this procedure is much easier to deal with, the present derivation of the equations (2.25) yields some more information, and seems to be more consistent with the development of an error analysis. One of the consequences of the system (2.25) is the possibility of deriving the kernels $K(x, t)$ and $K^*(x, t)$ as defined in (1.13) and all their properties from the system (2.25) together with (2.3) and (2.20). This, however, is beyond the purpose of this paper.

3. Error Bounds

For the purpose of obtaining bounds on $z(x)$ and $z^*(x)$ we make use of the fact that a uniform upper bound for the right-hand members of (2.25) is also a uniform upper bound for the solution. For the sake of simplicity this will be shown for the system (2.25.a) only.

The system reads

$$\begin{aligned}&\lambda_j z'_{j-1}+2z'_j+\mu_j z'_{j+1}=R_j\,, \qquad 1\leqq j\leqq n-1\\ &z'_0=z'_n=0\,.\end{aligned} \tag{3.1}$$

Let $|z'_i|=\max\limits_{1\leqq j\leqq n-1}|z'_j|$, then the following inequality holds:

$$|2z'_i|=|R_i-\lambda_i z'_{i-1}-\mu_i z'_{i+1}|\leqq|R_i|+|z'_i| \tag{3.2}$$

since $\lambda_i+\mu_i=1$. It follows

$$|z'_i|\leqq|R_i|\,. \tag{3.3}$$

Now let $M=\max\limits_{1\leqq j\leqq n-1}|R_j|$, then

$$|z'_j|\leqq M\,, \qquad 1\leqq j\leqq n-1\,. \tag{3.4}$$

In the same way we have

$$|z_j^{*\prime}|\leqq M^*\,, \qquad 0\leqq j\leqq n$$

where $M^*=\max\limits_{0\leqq j\leqq n}|R_j|$. (3.5)

In (2.25) R_j is given by an integral, which can be rearranged in the following form

$$R_j=\tfrac{1}{2}\frac{h_j h_{j+1}}{h_j+h_{j+1}}\int_0^1(1-t)^2t\{h_{j+1}^2 f^{\mathrm{IV}}(x_j+h_{j+1}t)-h_j^2 f^{\mathrm{IV}}(x_j-h_jt)\}\,dt$$

for $1\leqq j\leqq n-1$

$$\begin{aligned}R_0&=\ \tfrac{1}{2}h_1^3\int_0^1(1-t)^2tf^{\mathrm{IV}}(x_0+h_1t)\,dt\\ R_n&=-\tfrac{1}{2}h_n^3\int_0^1(1-t)^2tf^{\mathrm{IV}}(x_n-h_nt)\,dt\,.\end{aligned} \tag{3.6}$$

Now since $(1-t)^2t$ is a non-negative function it follows

$$\begin{aligned}|R_j|&\leqq\tfrac{1}{24}\frac{h_jh_{j+1}(h_j^2+h_{j+1}^2)}{h_j+h_{j+1}}\|f^{\mathrm{IV}}\|\,, \qquad 1\leqq j\leqq n-1\\ |R_0|&\leqq\tfrac{1}{24}\,h_1^3\|f^{\mathrm{IV}}\|\,.\\ |R_n|&\leqq\tfrac{1}{24}\,h_n^3\|f^{\mathrm{IV}}\|\,.\end{aligned} \tag{3.7}$$

Using the mesh norm, (3.7) can be simplified by

$$|R_j|\leqq\tfrac{1}{24}\|h\|^3\|f^{\mathrm{IV}}\|\,, \qquad 0\leqq j\leqq n\,. \tag{3.8}$$

Applying (3.4) we get

$$|z'_j|\leqq\tfrac{1}{24}\|h\|^3\|f^{\mathrm{IV}}\|\,, \qquad 1\leqq j\leqq n-1\,. \tag{3.9}$$

From (3.9) we can construct a bound on $z(x)$ by using the relation (2.7) and (2.10)

$$z(x)=\frac{(x-x_i)(x_{i+1}-x)}{h_{i+1}^2}\{z'_i(x_{i+1}-x)-z'_{i+1}(x-x_i)\}$$

for $x_i\leqq x\leqq x_{i+1}, 0\leqq i\leqq n-1$

from which it follows

$$|z(x)|\leqq\frac{(x-x_i)(x_{i+1}-x)}{24}\cdot\frac{\|h\|^3}{h_{i+1}} \tag{3.11}$$

for $x_i\leqq x\leqq x_{i+1},\ 0\leqq i\leqq n-1\,.$

Finally using the relations (2.3) and (2.2)

$$\begin{aligned} E_h(f;x) &= E_H(f;x)+z(x) \\ E_H(f;x) &= \frac{(x-x_i)^2(x_{i+1}-x)^2}{24} f^{\mathrm{IV}}(\xi_i) \end{aligned} \tag{3.12}$$

we arrive at the result

$$|E_h(f;x)| \leqq \frac{(x-x_i)(x_{i+1}-x)}{24}\left\{(x-x_i)(x_{i+1}-x)+\frac{\|h\|^3}{h_{i+1}}\right\}\|f^{\mathrm{IV}}\| \tag{3.13}$$

for $x_i \leqq x \leqq x_{i+1}$, $0 \leqq i \leqq n-1$.

The same procedure applied to $z^*(x)$ leads to the same result for $E_h^*(f;x)$:

$$|E_h^*(f;x)| \leqq \frac{(x-x_i)(x_{i+1}-x)}{24}\left\{(x-x_i)(x_{i+1}-x)+\frac{\|h\|^3}{h_{i+1}}\right\}\|f^{\mathrm{IV}}\|. \tag{3.14}$$

This last result is somewhat more specific than the result Atkinson obtained in [3] for $E_h^*(f;x)$. In fact his bounds for the integral in (1.15) can be improved upon in the following way:

Let ε be any positive number. Then for every fixed $x \in [x_i; x_{i+1}]$ a function f exists with $\|f^{\mathrm{IV}}\|=1$, so that:

$$\int_a^b |K^*(x,t)|\,dt-\varepsilon < |E_h^*(f;x)| \leqq \int_a^b |K^*(x,t)|\,dt. \tag{3.15}$$

It follows from (3.15) and (3.14):

$$\int_a^b |K^*(x,t)|\,dt \leqq \frac{(x-x_i)(x_{i+1}-x)}{24}\left\{(x-x_i)(x_{i+1}-x)+\frac{\|h\|^3}{h_{i+1}}\right\}. \tag{3.16}$$

Furthermore for any $\nu>1$ a function $g(x)$ exists, satisfying the relations:

$$g^{\mathrm{IV}}(x)=(-1)^j\left\{1-\left(\frac{x-x_j}{h_{j+1}}\right)^\nu-\left(\frac{x_{j+1}-x}{h_{j+1}}\right)^\nu\right\}, \quad x_j \leqq x \leqq x_{j+1}, \quad 0 \leqq j \leqq n-1. \tag{3.17}$$

Then $\|g^{\mathrm{IV}}\|=1$, and it follows from (3.6):

$$\begin{aligned} R_j &= (-1)^j|R_j|. \\ |R_j| &= \tfrac{1}{24}\frac{h_j h_{j+1}(h_j^2+h_{j+1}^2)}{h_j+h_{j+1}}(1-\varepsilon), \quad 1 \leqq j \leqq n-1 \\ |R_0| &= \frac{h_1^3}{24}(1-\varepsilon), \quad |R_n| = \frac{h_n^3}{24}(1-\varepsilon), \quad \varepsilon = \frac{12}{(\nu+2)(\nu+3)}. \end{aligned} \tag{3.18}$$

Introducing the mesh ratio β as follows:

$$\beta = \max_{1 \leqq j \leqq n} \frac{\|h\|}{h_j} \tag{3.19}$$

we can write

$$\begin{aligned} R_j &= (-1)^j|R_j| \\ m &\leqq |R_j| \leqq M \end{aligned} \tag{3.20}$$

with

$$M = \frac{\|h\|^3}{24}(1-\varepsilon) = \frac{\|h\|^3}{24}\left\{1-\frac{12}{(\nu+2)(\nu+3)}\right\}, \quad m = \frac{M}{\beta^3}$$

Now we can express $z_j^{*\prime}$ as follows

$$z_j^{*\prime}=(-1)^j(M-\delta_j), \qquad 0 \leqq \delta_j \leqq 2M \tag{3.21}$$

and applying the same reasoning as in the beginning of this chapter we get

$$|\delta_j| = \delta_j \leqq \max_j (M - |R_j|) = M - m\,. \tag{3.22}$$

Hence the $z_j^{*\prime}$ satisfy the relations:

$$\begin{aligned} z_j^{*\prime} &= (-1)^j |z_j^{*\prime}|\,, \\ m &\leqq |z_j^{*\prime}| \leqq M \end{aligned} \qquad 0 \leqq j \leqq n\,. \tag{3.23}$$

Hence the error $E_h^*(g;x)$ can be expressed as follows

$$|E_h^*(g;x)| = (x-x_i)(x_{i+1}-x)\left\{\frac{(x-x_i)(x_{i+1}-x)}{24} + \frac{m+\theta_i(M-m)}{h_{i+1}}\right\} \tag{3.24}$$

with $0 \leqq \theta_i \leqq 1$. So $|E_h^*(g;x)|$ has the following lower bound:

$$(x-x_i)(x_{i+1}-x)\left\{\frac{(x-x_i)(x_{i+1}-x)}{24} + \frac{m}{h_{i+1}}\right\} \leqq |E_h^*(g;x)|\,. \tag{3.25}$$

Since

$$|E_h^*(g;x)| \leqq \int_a^b |K^*(x,t)|\,dt\|g^{\mathrm{IV}}\| = \int_a^b |K^*(x,t)|\,dt$$

and using (3.20), we find if v tends to infinity:

$$\frac{(x-x_i)(x_{i+1}-x)}{24}\left\{(x-x_i)(x_{i+1}-x) + \frac{\|h\|^2}{\beta^3}\right\} \leqq \int_a^b |K^*(x,t)|\,dt\,. \tag{3.26}$$

Combining (3.16) and (3.26) we arrive at:

$$\begin{aligned} \frac{(x-x_i)(x_{i+1}-x)}{24}\left\{(x-x_i)(x_{i+1}-x) + \frac{\|h\|^2}{\beta^3}\right\} &\leqq \int_a^b |K^*(x,t)|\,dt \leqq \\ &\leqq \frac{(x-x_i)(x_{i+1}-x)}{24}\left\{(x-x_i)(x_{i+1}-x) + \beta\|h\|^2\right\} \end{aligned} \tag{3.27}$$

If the mesh is uniform, then $\beta = 1$, and we have exactly:

$$\int_a^b |K^*(x,t)|\,dt = \frac{(x-x_i)(x_{i+1}-x)}{24}\{(x-x_i)(x_{i+1}-x) + \|h\|^2\}\,. \tag{3.28}$$

4. Improvement of the Bounds on $E_h(f;x)$ if $f \in C^5[a,b]$

The error of optimal cubic spline interpolation can be more specifically bounded if $f \in C^5[a,b]$.

In that case an integration by parts can be applied to the integrals in (3.6)

$$\begin{aligned} R_j &= \tfrac{1}{2}\frac{h_j h_{j+1}}{h_j + h_{j+1}} \int_0^1 (1-t)^2 t\{h_{j+1}^2 f^{\mathrm{IV}}(x_j + h_{j+1}t) - h_j^2 f^{\mathrm{IV}}(x_j - h_j t)\}\,dt = \\ &= \tfrac{1}{2}\frac{h_j h_{j+1}}{h_j + h_{j+1}} \frac{h_{j+1}^2 - h_j^2}{12} f^{\mathrm{IV}}(x_j) + \\ &\quad + \int_0^1 \frac{(1-t)^3}{12}(1+3t)[h_{j+1}^3 f^{\mathrm{IV}}(x_j + h_{j+1}t) + h_j^3 f^{\mathrm{IV}}(x_j - h_j t)]\,dt \\ &= \frac{(h_{j+1}-h_j)h_{j+1}}{24} f^{\mathrm{IV}}(x_j) + \tfrac{1}{60}\frac{h_{j+1}^3 + h_j^3}{h_j + h_{j+1}} h_j h_{j+1} f^{\mathrm{V}}(\xi_j) \end{aligned} \tag{4.1}$$

with $x_{j-1} \leqq \xi_j \leqq x_{j+1}$, $1 \leqq j \leqq n-1$.

Introducing the local mesh norm

$$\|h_i\| = \max\{h_i, h_{i+1}\} \tag{4.2}$$

and the local mesh ratio:

$$\beta_i = \max\left\{\frac{\|h_i\|}{h_i}, \frac{\|h_i\|}{h_{i+1}}\right\} \tag{4.3}$$

then it follows

$$|h_{i+1}h_i(h_{i+1}-h_i)| = \frac{1}{\beta_i}\left(1-\frac{1}{\beta_i}\right)\|h_i\|^3 . \tag{4.4}$$

Now if the mesh is uniform, β_i equals 1 and hence we may regard $1-1/\beta_i$ as a local measure for nonuniformity

$$1-\frac{1}{\beta_i} = \varepsilon_i, \qquad 0 \leqq \varepsilon_i \leqq 1 . \tag{4.5}$$

If the local nonuniformity is bounded by:

$$\varepsilon_i < \varepsilon, \quad \text{for } 1 \leqq i \leqq n-1 \tag{4.6}$$

then

$$\begin{aligned} &\varepsilon_i(1-\varepsilon_i) \leqq \tfrac{1}{4} \quad \text{if } \varepsilon \geqq \tfrac{1}{2} \\ &\varepsilon_i(1-\varepsilon_i) \leqq \varepsilon(1-\varepsilon) \quad \text{if } \varepsilon \leqq \tfrac{1}{2} . \end{aligned} \tag{4.7}$$

Define the monotonic nondecreasing function $\vartheta(\varepsilon)$ as

$$\vartheta(\varepsilon) = \begin{cases} 1, & \tfrac{1}{2} \leqq \varepsilon \leqq 1 \\ 4\varepsilon(1-\varepsilon), & 0 \leqq \varepsilon \leqq \tfrac{1}{2} \end{cases} \tag{4.8}$$

then it follows:

$$|h_i h_{i+1}(h_{i+1}-h_i)| \leqq \tfrac{1}{4}\vartheta(\varepsilon)\|h_i\|^3 \leqq \tfrac{1}{4}\vartheta(\varepsilon)\|h\|^3 . \tag{4.9}$$

The expression (4.1) for R_j can now be bounded:

$$|R_j| \leqq \frac{\vartheta(\varepsilon)}{96}\|h\|^3\|f^{\mathrm{IV}}\| + \tfrac{1}{60}\|h\|^4\|f^{\mathrm{V}}\|) . \tag{4.10}$$

If the mesh is nearly uniform, the first term of (4.10) will be very small; quantitatively we can write

$$|R_j| = O(\varepsilon\|h\|^3\|f^{\mathrm{IV}}\|) + O(\|h\|^4\|f^{\mathrm{V}}\|) . \tag{4.11}$$

If $\varepsilon = O(\|h\|^2)$

$$|R_j| \leqq \tfrac{1}{60}\|h\|^4\|f^{\mathrm{V}}\| + O(\|h\|^5\|f^{\mathrm{IV}}\|) . \tag{4.12}$$

The result (4.12) has a bearing on parametric spline interpolation of smooth curves, where only nearly uniform meshes can be constructed if uniform meshes are required.

Applying (3.4) we get:

$$|z_j'| \leqq \frac{\vartheta(\varepsilon)}{96}\|h\|^3\|f^{\mathrm{IV}}\| + \tfrac{1}{60}\|h\|^4\|f^{\mathrm{V}}\| . \tag{4.13}$$

and according to (2.6) it follows:

$$|E_h'(f; x_j)| \leqq \frac{\vartheta(\varepsilon)}{96}\|h\|^3\|f^{\mathrm{IV}}\| + \tfrac{1}{60}\|h\|^4\|f^{\mathrm{V}}\| . \tag{4.14}$$

This result is an extension of the statement

$$|E_h'(f; x_j)| = O(\|h\|^4) \tag{4.15}$$

for nearly uniform meshes.

Finally by applying (3.10) and (3.12) we arrive at the following bound for $E_h(f; x)$:

$$|E_h(f;x)| \leqq (x-x_i)(x_{i+1}-x)\|h\|^2 \left\{\frac{\|f^{\mathrm{IV}}\|}{96}\left[1+\vartheta(\varepsilon)\frac{\|h\|}{h_{i+1}}\right]+\frac{\|f^{\mathrm{V}}\|}{60}\frac{\|h\|^2}{h_{i+1}}\right\},$$
$$x_i \leqq x \leqq x_{i+1} \quad (4.16)$$

which bound is much more specific than that given by (3.14). For uniform meshes the expression (4.16) reads:

$$|E_h(f;x)| \leqq (x-x_i)(x_{i+1}-x)\|h\|^2 \left\{\frac{\|f^{\mathrm{IV}}\|}{96}+\frac{\|f^{\mathrm{V}}\|\,\|h\|}{60}\right\}, \qquad x_i \leqq x \leqq x_{i+1}. \quad (4.17)$$

Further refinements of the error analysis can be made by examining the equations (2.25) thoroughly; this, however, is beyond the purpose of this paper.

Extensions of the theory are possible in cases of periodic splines, which occur in approximation of periodic functions and in parametric spline interpolation of closed smooth curves.

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